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Compton scattering in a converging fluid flow – I. The transfer equation

R. D. Blandford and D. G. Payne *W. K. Kellogg Radiation Laboratory, California Institute of Technology, Pasadena, California 91125, USA*

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Summary. The equations describing Compton scattering in an optically thick fluid flow are derived under the diffusion approximation. The relative importance of the bulk and random motions of the scattering electrons is discussed. In a converging fluid flow of speed u, bulk acceleration of photons dominates thermal Comptonization when $u \gtrsim (12\, kT_e/m_e)^{1/2}$.

1 Introduction

From observations of O stars, X-ray binaries, pulsars and quasars, it has become apparent that in many astronomical souces, radiation is emitted by hot plasmas that are too tenuous for local thermodynamic equilibrium to be established but are nevertheless quite optically thick to Thomson scattering by free electrons. Under these conditions, the emergent spectrum can be markedly non-thermal (e.g. Kompaneets 1957; Illarionov & Sunyaev 1972; Katz 1976; Shapiro, Lightman & Eardley 1976).

In most existing treatments of Comptonization, it has been assumed that the plasma forms a static atmosphere and that successive scatterings, on average, either blueshift the photons (when the electron temperature T_e exceeds the photon energy) or redshifts them (when the inequality is reversed). (When the radiation brightness temperature exceeds the electron rest mass, induced scattering is important: Goldreich, McCray & Rees 1968; Zel'dovich & Sunyaev 1972.) In particular, soft photons diffusing through a hot electron gas can be accelerated so that an exponentially small fraction are scattered many times to an exponentially large frequency, thus generating a power-law spectrum. Within the escape probability formalism, the spectral index α satisfies

$$\alpha(\alpha+3) = \left(\frac{kT_e}{m_e c^2}\right)^{-1} \tau_T^{-2}, \tag{1}$$

where τ_T is the Thomson optical depth (Eardley & Lightman 1976; Blandford & Rees 1978; Payne 1980; Sunyaev & Titarchuk 1980). Reflection from a half-plane produces an approximately flat spectrum with $\alpha \sim 0$ (Lightman & Rybicki 1979).

However, in many of these environments the scattering medium is likely to be undergoing large bulk motions. In a non-uniform fluid flow (i.e. converging or diverging) of characteristic speed u, the photons will be preferentially accelerated by the bulk motion if the Thomson optical depth of the fluid is at least (c/u) and the electron thermal velocity is less than u.

In this paper we present a formal derivation of the transfer equation obeyed by the photon occupation number, under the diffusion approximation. (Analogous results have been derived in the context of cosmic-ray propagation theory, e.g. Skilling 1975.) From this equation we can gauge the relative importance of thermal Comptonization and bulk acceleration. In two forthcoming papers we consider applications of our results to a radiation-dominated shock, in the limit of ignorable thermal effects and fresh photon production, and a super-critical spherical accretion flow.

2 The transfer equation

Consider a fully ionized, non-relativistic plasma with an electron temperature T_e and an electron density n_e, which is undergoing some non-relativistic bulk motion characterized by a velocity $\mathbf{u}$. The plasma is assumed to be optically thick to electron scattering and optically thin to free–free absorption.

The transfer equation we derive is appropriate for an observer in a fixed inertial frame. However, since the scattering integrals take their simplest form in the rest frame of the electrons, we evaluate them in that frame. Throughout the rest of this paper we will use a subscript 'o' to denote a quantity measured in the electron rest frame.

The radiation field is characterized by the photon occupation number $n(\mathbf{x}, \mathbf{k})$, which is a scalar

$$n(\mathbf{x}, \mathbf{k}) = n(\mathbf{x}_o, \mathbf{k}_o). \tag{2}$$

In equation (2), $\mathbf{x}$ gives the photon's position in space-time, while $\mathbf{k}$ gives its four-momentum. As measured in an inertial frame,

$$k^\mu = \nu(1, \mathbf{l}), \tag{3}$$

where ν is the photon's frequency and $\mathbf{l}$ is a unit vector in the photon's direction of motion (throughout this paper we work in units in which $h = k = c = 1$).

The manifestly covariant equation of radiative transfer (e.g. Lindquist 1966; Thorne 1981) is, in Lindquist's notation,

$$k^\mu \frac{Dn}{dx^\mu} = \int R(\mathbf{k}', \mathbf{k})\, n(\mathbf{k}')\, dK' - \int R(\mathbf{k}, \mathbf{k}')\, n(\mathbf{k})\, dK' + g, \tag{4}$$

where

$k^\mu \dfrac{D}{dx^\mu}$ = directional derivative along a photon trajectory in phase space,

R = photon redistribution function (per unit length),

$dK = \nu d\nu\, d\Omega = \nu_o d\nu_o\, d\Omega_o$

= invariant volume element in photon phase space,

$d\Omega$ = element of solid angle,

g = invariant photon source function.

In equation (4) we have assumed that stimulated scattering and absorption are negligible. If the transfer equation is studied in a non-inertial frame (i.e. the comoving frame), then coordinate acceleration terms arise on the left-hand side of equation (4), while the scattering terms take a simple form. This is the approach used by Castor (1972). If we study the transfer equation in an inertial frame, then the left-hand side of equation (4) takes its simplest form, while the scattering terms become more complicated due to the coupling between angle and frequency arising from the Doppler and aberration effects. Hsieh & Spiegel (1976) have studied the transfer equation using this approach. The choice of frame used depends on the physical context of the problem. It turns out here that the two methods lead to an identical transfer equation to the order to which we are working.

In the electron rest frame,

$$k^{\mu}\frac{Dn}{dx^{\mu}}=\frac{3}{16\pi}n_{e_o}\nu_o\int d\Omega_o'[1+(\mathbf{l}_o\cdot\mathbf{l}_o')^2]\sigma(\nu_o+\delta\nu_o)n(\mathbf{l}_o',\nu_o+\delta\nu_o)\left(\frac{\nu_o+\delta\nu_o}{\nu_o}\right)^2$$

$$\times\frac{\partial}{\partial\nu_o}(\nu_o+\delta\nu_o)-n_{e_o}\nu_o\sigma(\nu_o)n(\mathbf{l}_o,\nu_o)+\nu_o j_o, \qquad (5)$$

where

$$\delta\nu_o\equiv\nu_o^2/m_e(1-\mathbf{l}_o\cdot\mathbf{l}_o'). \qquad (6)$$

The term $(\nu_o+\delta\nu_o/\nu_o)^2\,\partial/\partial\nu_o\,(\nu_o+\delta\nu_o)$ takes into account the larger phase space at the higher frequency $(\nu_o+\delta\nu_o)$, and we have introduced

$$g(\nu_o)=\nu_o j_o, \qquad (7)$$

where j_o is the photon source function in phase space, as measured in the electron rest frame. Finally, for generality, we have allowed for the possibility of an energy-dependent cross-section by writing

$$\frac{d\sigma}{d\Omega}=\frac{3}{16\pi}\sigma(\nu)[1+(\mathbf{l}\cdot\mathbf{l}')^2]. \qquad (8)$$

Now,

$$n(\mathbf{l}_o',\nu_o+\delta\nu_o)=n\left(\mathbf{l}',\nu\left[\frac{1+\mathbf{l}_o'\cdot\mathbf{u}}{1+\mathbf{l}_o\cdot\mathbf{u}}\right]\left[1+\frac{\delta\nu_o}{\nu_o}\right]\right)$$

$$\simeq n(\mathbf{l}',\nu)+[(\mathbf{l}_o'-\mathbf{l}_o)\cdot\mathbf{u}+(\mathbf{l}_o\cdot\mathbf{u})(\mathbf{l}_o-\mathbf{l}_o')\cdot\mathbf{u}+\delta\nu_o/\nu_o]\,\nu\frac{\partial}{\partial\nu}n(\mathbf{l}',\nu)$$

$$+\tfrac{1}{2}[(\mathbf{l}_o'-\mathbf{l}_o)\cdot\mathbf{u}]^2\nu^2\frac{\partial^2}{\partial\nu^2}n(\mathbf{l}',\nu), \qquad (9)$$

to second order in (u/c) and first order in $(h\nu_o/m_e c^2)$, while

$$\sigma(\nu_o+\delta\nu_o)=\sigma(\nu_o)+(\delta\nu_o/\nu_o)\,\nu\frac{\partial}{\partial\nu}\sigma(\nu_o), \qquad (10)$$

to first order in $(h\nu_o/m_e c^2)$. In a stationary, inertial frame the transfer equation then becomes

$$\nu\left[\frac{\partial}{\partial t}n(\mathbf{l},\nu)+\mathbf{l}\cdot\nabla n(\mathbf{l},\nu)\right]=\frac{3}{16\pi}n_{e_o}\nu_o\sigma(\nu_o)\int d\Omega'_o[1+(\mathbf{l}_o\cdot\mathbf{l}'_o)^2]$$
$$\times\left\{n(\mathbf{l}',\nu)+[(\mathbf{l}'_o-\mathbf{l}_o)\cdot\mathbf{u}+(\mathbf{l}_o\cdot\mathbf{u})(\mathbf{l}_o-\mathbf{l}'_o)\cdot\mathbf{u}]\nu\frac{\partial}{\partial\nu}n(\mathbf{l}',\nu)\right.$$
$$+\tfrac{1}{2}[(\mathbf{l}'_o-\mathbf{l}_o)\cdot\mathbf{u}]^2\nu^2\frac{\partial^2}{\partial\nu^2}n(\mathbf{l}',\nu)+\frac{\nu}{m_e}(1-\mathbf{l}_o\cdot\mathbf{l}'_o)\left[4n(\mathbf{l}',\nu)\right.$$
$$\left.\left.+\nu\frac{\partial}{\partial\nu}n(\mathbf{l}',\nu)+\left(\frac{\partial\ln\sigma(\nu)}{\partial\ln\nu}\right)n(\mathbf{l}',\nu)\right]\right\}-n_{e_o}\nu_o\sigma(\nu_o)n(\mathbf{l},\nu)+\nu_o j_o. \tag{11}$$

The right-hand side of the transfer equation now involves only the various moments of the radiation field and their derivatives. In solving the transfer equation by the moment method, it is necessary to close the system of equations. For the problems we are interested in, the radiative transfer can be treated as a diffusion process. Under the usual diffusion approximation we have,

$$n(\mathbf{l},\nu)=\bar{n}(\nu)+3\mathbf{l}\cdot\mathbf{f}(\nu), \tag{12}$$

where

$$\bar{n}(\nu)=\frac{1}{4\pi}\int d\Omega\, n(\mathbf{l},\nu), \tag{13}$$

and

$$\mathbf{f}(\nu)=\frac{1}{4\pi}\int d\Omega\,\mathbf{l}\,n(\mathbf{l},\nu). \tag{14}$$

Expanding $\sigma(\nu_o)$ in equation (11) to second order in (u/c) in a manner similar to that done for $n(\mathbf{l}'_o,\nu'_o)$, transforming $n_{e_o}\nu_o$ to the inertial frame, and making use of the closure approximation of equation (12), we can obtain the moments of the transfer equation. The first and second moments are respectively,

$$\frac{\partial\bar{n}}{\partial t}+\nabla\cdot\mathbf{f}=n_e\sigma(\nu)\left\{\frac{1}{3}u^2\left[4\nu\frac{\partial\bar{n}}{\partial\nu}+\nu^2\frac{\partial^2\bar{n}}{\partial\nu^2}+\left(\frac{\partial\ln\sigma}{\partial\ln\nu}\right)\nu\frac{\partial\bar{n}}{\partial\nu}\right]+\nu\frac{\partial}{\partial\nu}(\mathbf{u}\cdot\mathbf{f})\right.$$
$$\left.+\left(\frac{\partial\ln\sigma}{\partial\ln\nu}\right)(\mathbf{u}\cdot\mathbf{f})+\frac{\nu}{m_e}\left[4\bar{n}+\nu\frac{\partial\bar{n}}{\partial\nu}+\left(\frac{\partial\ln\sigma}{\partial\ln\nu}\right)\bar{n}\right]\right\}+\bar{j}, \tag{15}$$

and

$$\frac{1}{3}\nabla\bar{n}=n_e\sigma(\nu)\left(-\frac{1}{3}\mathbf{u}\nu\frac{\partial\bar{n}}{\partial\nu}-\mathbf{f}\right). \tag{16}$$

In equation (15),

$$\bar{j}(\nu)=\frac{1}{4\pi}\int d\Omega\, j(\nu,\mathbf{l}). \tag{17}$$

$$\langle \nu \rangle \equiv \frac{\int d\nu\, \nu^4 \bar{n}(\nu)}{\int d\nu\, \nu^3 \bar{n}(\nu)}, \tag{26}$$

S = photon emissivity.

σ_R given by equation (24) is just the Rosseland mean cross-section, while σ_K given by equation (25) is analogous to the Planck mean. For Compton scattering $\sigma_R = \sigma_K = \sigma_T$, of course. $\langle \nu \rangle$ is the energy-weighted mean photon frequency. An additional mean frequency which is of interest is the number-weighted mean photon frequency

$$\bar{\nu} \equiv \frac{\int d\nu\, \nu^3 \bar{n}(\nu)}{\int d\nu\, \nu^2 \bar{n}(\nu)}. \tag{27}$$

4 Discussion of competitive processes

Under many astrophysical conditions, thermal effects are likely to be at least as important as bulk acceleration and equations (18) and (22) must be solved in their entirety, presumably numerically.

The characteristic time-scales for the spectrum to be influenced by Compton heating, Compton cooling, and compressional heating (or equivalently, expansion cooling) are given respectively by

$$t_h^{-1} \sim n_e \sigma_T c \left(\frac{4kT_e}{m_e c^2}\right), \tag{28}$$

$$t_c^{-1} \sim n_e \sigma_T c \left(\frac{h\nu}{m_e c^2}\right), \tag{29}$$

$$t_u^{-1} \sim (1/3\, \nabla \cdot \mathbf{u}) \sim n_e \sigma_T c \left(\frac{1}{3}\frac{u^2}{c^2}\right), \tag{30}$$

where we have assumed the velocity scale-length to be $\sim c/(n_e \sigma_T u)$, as is typical (*cf.* Blandford & Payne 1981a, b – Papers II and III). Hence, for $u \gtrsim (12\, kT_e/m_e)^{1/2}$, bulk acceleration is more important than thermal Comptonization. For comparison, the rate for electrons to achieve Maxwellian equilibrium at a temperature T_e is typically

$$t_e^{-1} \sim n_e \sigma_T c (kT_e/m_e c^2)^{-3/2} \ln \lambda, \tag{31}$$

where $\ln \lambda$ is the usual Coulomb logarithm (e.g Spitzer 1978). For the non-relativistic flows we are considering, this is always faster than t_u^{-1} and so we can safely assume that the electrons are locally Maxwellian. In fact, there is also generally time for the electrons to come into equilibrium with the ions, which takes a factor $\sim (m_p/m_e)$ longer.

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